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ESTIMATION OF UNKNOWN BOUNDARY VALUES FROM BOUNDARY AND INSIDE OBSERVATIONS

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Abstract – Some inverse boundary values problems deal with the estimation of boundary conditions on incompletely prescribed boundaries. The boundary values on incompletely prescribed boundaries can be estimated when excessively prescribed boundaries are introduced. Inside observations may be used together with the boundary observations for estimating the unknown boundary values. In this paper the inside displacement observations are used together with boundary observations to estimate the unknown boundary conditions for an elastostatic body. By applying the boundary element equations, these observations were incorporated in a matrix equation for unknown boundary values. The unknown boundary values may be estimated by solving the matrix equation. The matrix equation is ill-conditioned due to the ill-posed nature of the problem, and regularization is necessary to obtain a good solution. An alternating boundary element inverse analysis scheme is developed for the estimation from the boundary and inside observations. As the regularization parameter, the stopping number of iterations is employed. Its reasonable estimation is made by using the discrepancy principle applied in observations or in observation equations. It is found that the iterative boundary element inverse analysis scheme is useful for the estimation of boundary values from the inside and boundary observations.

1. INTRODUCTION

There are various kinds of inverse boundary value problems in science and engineering [1]. Inverse boundary values problems deal with the estimation of boundary conditions on incompletely prescribed boundaries, where boundary conditions are incompletely prescribed [1-17]. The boundary values on incompletely prescribed boundaries can be estimated, when excessively prescribed boundaries are introduced [8, 12, 15]. By applying the boundary-element method [18, 19] the inverse boundary value problem is reduced to the solution of a matrix equation. The matrix equation can be solved for unknown boundary values.

For two-dimensional elastostatic problems, measurements concerning displacement and strains can be made at points inside the body. These observations may be used for estimating the unknown boundary values. By applying the boundary element equations, these measurements can be used for constructing matrix equations for unknown boundary values. In this case again the unknown boundary values may be estimated by solving the matrix equation.

The matrix equation is severely ill-conditioned because of the ill-posed nature of the problem. When inverse analysis scheme without regularization is applied, errors included in values of the excessively prescribed boundaries are magnified tremendously in the estimated boundary values. Regularization is therefore necessary to obtain a good or reasonable solution of this matrix equation for the estimated boundary values. The present authors applied the singular value decomposition with rank reduction to solve the ill-posed matrix [17]. The rank was estimated reasonably with the discrepancy principle [20]. The alternating boundary element inverse analysis scheme was proposed by Kolzov *et al.* [7]. They discussed the convergence of solution obtained by using the scheme. Lesnic *et al.* [9, 10, 13] and Kubo and Furukawa [14] examined the applicability of the scheme to the identification of boundary values for the Laplace field. Kubo *et al.* [16] applied the method to the estimation of boundary values from noisy observations. In their study the number of iterations at truncation was regarded as a regularization parameter and was estimated by using the discrepancy principle [20].

In the present paper the alternating boundary element inversion analysis scheme is developed for estimating unknown boundary values from inside observations together with boundary observations on over-prescribed boundary values. Numerical simulations are made to examine the applicability of the scheme. Selection of regularization parameter, i.e. the stopping number of iterations, is discussed.

2. BASIC EQUATIONS FOR SOLVING INVERSE BOUNDARY VALUE PROBLEM WITH INSIDE AND BOUNDARY OBSERVATIONS

In direct problems, boundary value is prescribed at every point of boundary of a body. The inverse boundary value problem involves incompletely prescribed boundary, where no information concerning the boundary values is available in advance. For the estimation of the incompletely prescribed boundary, excessively prescribed boundaries are introduced [2, 8, 11, 15].

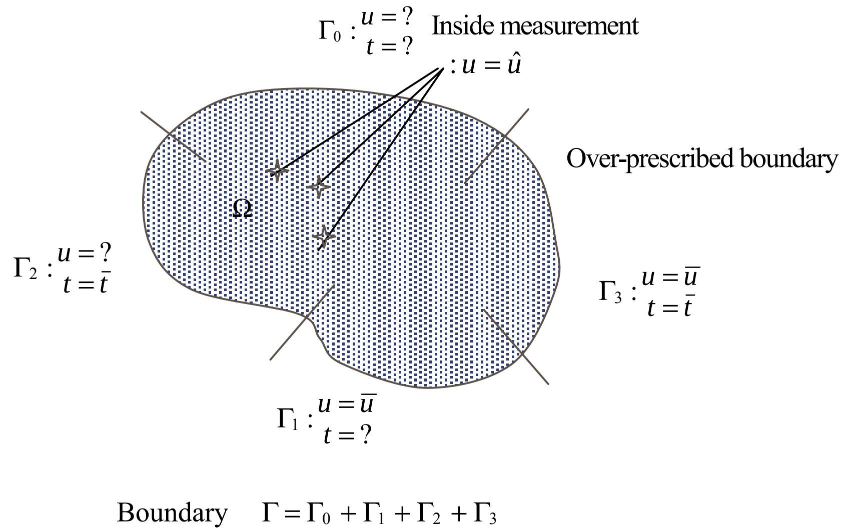

Figure 1: Boundary conditions for elastostatic inverse boundary value problem.

Consider an inverse boundary value problem of elastostatics shown in Figure 1. For boundary value problem of elastostatics, the boundary integral equation is obtained by integrating the Navier's equations multiplied by the fundamental solutions u_{ij}*, t_{ij}* [18, 19]. The displacement at point P inside the domain can be expressed as an integral along boundary $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3$ of the domain as

$$u_j(\mathrm{P}) = \int_\Gamma u_{ij}^*(x,\mathrm{P}) t_i(x) d\Gamma(x) - \int_\Gamma t_{ij}^*(x,\mathrm{P}) u_i(x) d\Gamma(x). \tag{1}$$

As the fundamental solutions u_{ij}*, t_{ij}*, Kelvin's solutions were used in this study. When the point P is moved to the boundary point Q, eqn. (1) is reduced to the following equation:

$$c_j u_j(\mathrm{Q}) = \mathrm{p.v.} \int_\Gamma u_{ij}^*(x,\mathrm{Q}) t_i(x) d\Gamma(x) - \int_\Gamma t_{ij}^*(x,\mathrm{Q}) u_i(x) d\Gamma(x). \tag{2}$$

Here, "p.v." designates the principal value of the integral and c_j is a coefficient dependent on the geometry near the point Q. By discretizing the boundary into boundary elements and applying the boundary-element method, the boundary integral eqn (2) is reduced to the following matrix equation, which interrelates vector $\{\boldsymbol{u}\}$ consisting of displacements at nodal points and vector $\{\boldsymbol{t}\}$ consisting of nodal displacements:

$$[\boldsymbol{H}]\{\boldsymbol{u}\} = [\boldsymbol{G}]\{\boldsymbol{t}\}. \tag{3}$$

For a direct elastostatic problem, boundary conditions concerning displacement or traction are given for each direction x_i at every point of boundary Γ. Then half of the boundary displacements and tractions at nodal points are known in advance. The rest of the boundary values are unknown and can be computed using eqn. (3) from prescribed boundary values. By moving unknown boundary values in eqn. (3) to the left side and prescribed boundary values in eqn. (3) to the right side, a matrix equation for the unknown boundary values is obtained:

$$[\boldsymbol{A}]\{\boldsymbol{x}\} = [\boldsymbol{B}]\{\boldsymbol{b}\} \tag{4}$$

where $\{\boldsymbol{x}\}$ is unknown boundary value vector and $\{\boldsymbol{b}\}$ is prescribed boundary value vector. Matrix $[\boldsymbol{A}]$ in eqn. (4) for this case is square and eqn. (4) can be solved without difficulties.

Inverse boundary value problems involve incompletely prescribed boundary, where no boundary conditions are prescribed. When both the traction and the displacement are prescribed on some parts of boundary, the incompletely prescribed boundary values may be estimated. In this case eqn. (3) can also be used to construct eqn. (4), which was used in the foregoing papers [2, 8, 11, 15]. In this case, however, there is no guarantee that matrix $[\boldsymbol{A}]$ in eqn. (4) is square. Matrix $[\boldsymbol{A}]$ is singular or nearly singular due to the ill-posed nature of the inverse problem.

When inside observations of displacements are made, we can use eqn. (1) which involves unknown boundary values. When inside observations of strains instead of displacements are made, we can use the following equation deduced by differentiating eqn. (1).

$$u_{j,k}(\mathrm{A}) = \int_\Gamma u^*_{ij,k}(x,\mathrm{A})t_i(x)d\Gamma(x) - \int_\Gamma t^*_{ij,k}(x,\mathrm{A})u_i(x)d\Gamma(x) \tag{5}$$

Here ",k" denotes differentiation with x_k. Eqns (1) and (5) are reduced to the following system of equations for the unknown boundary values.

$$\{\boldsymbol{c}\}+[\boldsymbol{h}]\{\boldsymbol{u}\} = [\boldsymbol{g}]\{\boldsymbol{t}\} \tag{6}$$

Here $\{\boldsymbol{c}\}$ denotes a vector consisting of measured displacements and strains. Then eqns (4) and (6) give the following equation with $\{\boldsymbol{d}\}$ denoting a vector calculated from prescribed boundary values and measured inside displacements and strains $\{\boldsymbol{c}\}$ and matrices $[\boldsymbol{H}]$, $[\boldsymbol{G}]$, $[\boldsymbol{h}]$ and $[\boldsymbol{g}]$.

$$[\boldsymbol{A}]\{\boldsymbol{x}\} = \{\boldsymbol{d}\} \tag{7}$$

This equation may be solved for the unknown boundary values.

When observations are made at a finite number of points, the uniqueness of the continuous problem does not hold. The discretization of the problem can recover the uniqueness, but stability of the solution is not assured and then the regularization is necessary for obtaining a reasonable solution. In the present discretized problem, matrix $[\boldsymbol{A}]$ is singular or nearly singular, and eqn. (7) does not give an approximate solution without regularization.

3. ALTERNATING BOUNDARY ELEMENT INVERSE ANALYSIS SCHEME MODIFIED FOR BOUNDARY VALUE ESTIMATION FROM INSIDE AND BOUNDARY OBSERVATIONS

As an inverse analysis scheme with regularization, the alternating boundary element inverse analysis scheme [7] was proposed for inverse boundary value problems where the boundary values are estimated from over-prescribed boundary values. In this section the method is modified for the case where the inside observations can be used together with the over-prescribed boundary values.

As an example consider an inverse elastostatics boundary value problem for a rectangular region of aspect ratio h shown in Figure 2 is considered. The boundary AD is regarded as the incompletely-prescribed boundary, where neither the displacements nor the tractions are prescribed. A part of boundary BC is taken as the over-prescribed boundary, while ordinary boundary conditions are prescribed on the rest part of the boundary BC. The inside points for measurement are placed on the line of EF.

The alternating boundary element inversion scheme consists of the following steps.

[Step 1] On the over-prescribed boundary the Dirichlet type displacement boundary condition is employed. Give an initial guess of the Neumann type traction boundary condition on the incompletely-prescribed boundary. On the rest of whole boundary the prescribed boundary condition is used. Inside observations are given on inside observation points. Solve boundary element eqns (4) and (6) simultaneously for unknown boundary values.

[Step 2] On the over-prescribed boundary the Neumann-type traction boundary condition is used. On the incompletely-prescribed boundary the Dirichlet-type displacement boundary condition obtained in the preceding step is used. On the rest of whole boundary the prescribed boundary condition is used. Inside observations are given on inside observation points. Solve boundary element eqns (4) and (6) simultaneously for unknown boundary values.

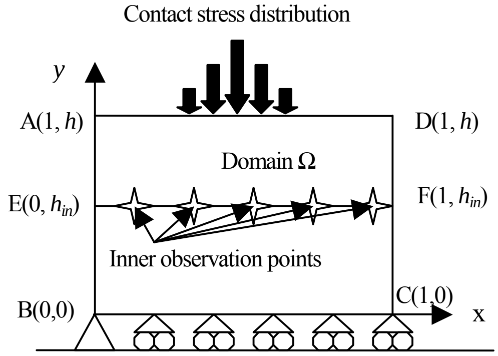

Figure 2: Boundary conditions for an inverse elastostatic boundary value problem.

[Step 3] On the over-prescribed boundary the Dirichlet-type displacement boundary condition is used. On the incompletely-prescribed boundary the Neumann-type traction boundary condition obtained in the preceding step is used. On the rest of whole boundary the prescribed boundary condition is used. Inside observations are given on inside observation points. Solve boundary element eqns (4) and (6) simultaneously for unknown boundary values.

[Step 4] Iterate the boundary element analyses as in Steps 2 and 3.

4. SELECTION OF NUMBER OF ITERATIONS AS A REGURARIZATION PARAMAETER

When the alternating inverse boundary element method is applied, the stopping number of iterations can be taken as a regularization parameter, as was discussed in [16]. The discrepancy principle [20] can be applied to determine the stopping number of iterations. In the conventional discrepancy principle [20] the discrepancy is evaluated between the given response and that corresponding to the solution. The regularization parameter, which gives a discrepancy of the observation noise level, is taken as an appropriate one.

The present authors proposed a discrepancy principle, which evaluates the discrepancy in the observation equations [15]. In this study observation equations are given by eqns (4) and (6).

5. PROCEDURE OF NUMERICAL SIMULATIONS

A rectangular region of the aspect ratio h shown in Figure 2 is considered. Boundary AD is regarded as the incompletely-prescribed boundary. Partial boundary of BC was taken as the over-prescribed boundary. The inside points for measurement are placed on the line of $y = h_{in}$. In the numerical simulations $h = 0.5$ and $h_{in} = 0.4$ were used. In the boundary element discretization, the number of boundary elements n was taken to be 10 for each of the top boundary AD, the bottom boundary BC, and the side boundaries AB and CD. The total number of inside observations n_{in} and the number over-prescribed boundary values n_{over} is taken to agree with the number of nodes on the incompletely-prescribed boundary.

To generate the over-prescribed boundary values on BC, direct pre-analyses were made using the boundary element method for the case that Hertzian type contact stress distribution is applied on AD. In the direct pre-analysis traction t_x (= σ_x) and t_y (= τ_{xy}) are prescribed to be 0 and other boundary values are unknown on AB and CD, while on boundary BC displacement u_y and traction t_x (= τ_{xy}) are prescribed to be 0 and other boundary values are taken to be unknown. The calculated values of traction and displacements on the over-prescribed boundary and the displacements at the inside observation points are used in the simulations of inverse analysis. To simulate the effect of errors in the measured displacements, noise of the order of 0.1%, 1%, 5% and 100% was introduced in the calculated values.

In the inverse analysis the displacements at the inside observation points, the boundary values of traction and displacement on the over-prescribed boundary of part of BC, displacement $u_y = 0$ traction t_x (= τ_{xy}) = 0 on the rest of BC, together with traction t_x (= σ_x) = 0 and t_y (= τ_{xy}) = 0 on AB and CD were used.

6. RESULTS OF NUMERICAL SIMULATION OF INVERSE BOUNDARY VALUE ANALYSES

As an example Figure 3 shows the variation of square sum of residual in contact stress distribution with iterations for h = 0.5, h_{in} = 0.4, the total number of inside observations n_{in}=6, and the number over-prescribed boundary values n_{over}=4.

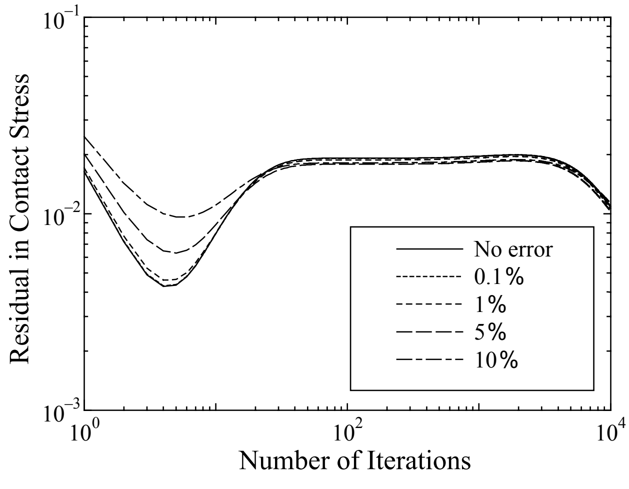

Figure 3: Variation of the square sum of residual in contact stress distribution with iterations (stress type:Hertzian, n=10, h=0.5, n_{in}=6, h_{in}=0.4, n_{over}=4).

The residual is evaluated between the actual and estimated contact stress distributions. As can be seen in the figure, in the first stage of iterations the residual in contact stress decreases with the number of iterations, then the residual increases with the number of iterations. This behavior implies that the solution gets better in the first stage and then after certain number of iterations it gets worse. There is an optimum number of the iterations. The number of iterations of truncation can be regarded as a regularization parameter.

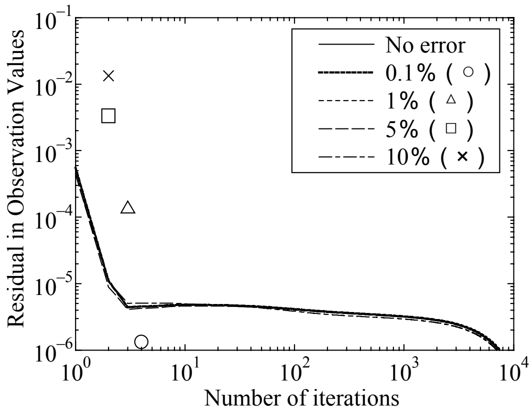

Figure 4: Variation of square sum of residual in observation values with iterations (stress type:Hertzian, n=10, h=0.5, n_{in}=6, h_{in}=0.4, n_{over}=4).

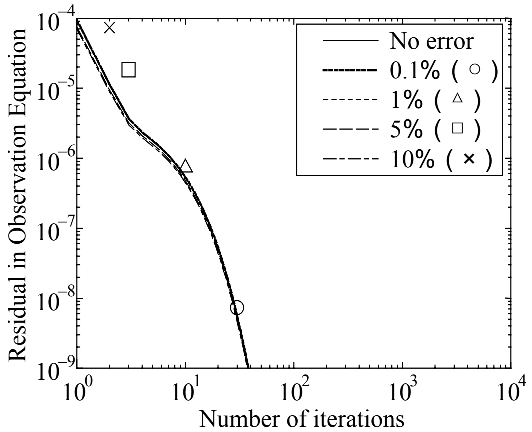

Figure 5: Variation of square sum of residual in observation equations with iterations (stress type:Hertzian, n=10, h=0.5, n_{in}=6, h_{in}=0.4, n_{over}=4).

To determine the optimum stopping number of iterations, the discrepancy principle applied in the ordinary observation space [20], and that applied in the observation equations [15] are used. The calculated residual in observation values and observation equations are shown in Figures 4 and 5, respectively. The residual in the observation values and that in the observation equations, which correspond to the noise level, are shown by symbols in the figures. By taking the value of discrepancy corresponding to the noise level, the optimum stopping number of iterations can be estimated from the figures.

From Figures 4 and 5 the stopping number of iterations estimated by the ordinary discrepancy principle applied in the observation space is smaller than that estimated by the discrepancy principle applied in the observation equations.

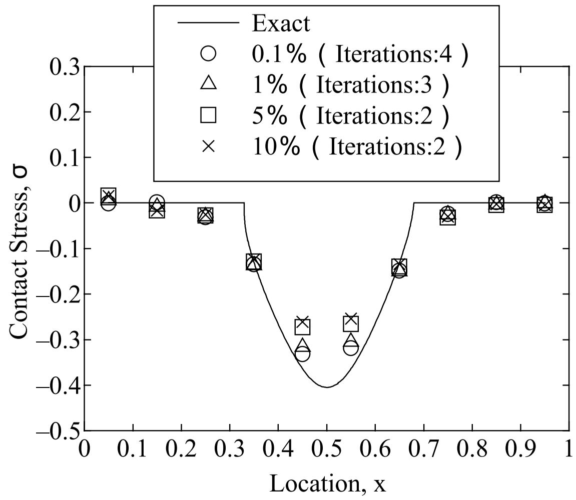

Figure 6: Contact stress distributions estimated by using discrepancy principle applied in observational space (stress type:Hertzian, n=10, h=0.5, n_{in}=6, h_{in}=0.4, n_{over}=4).

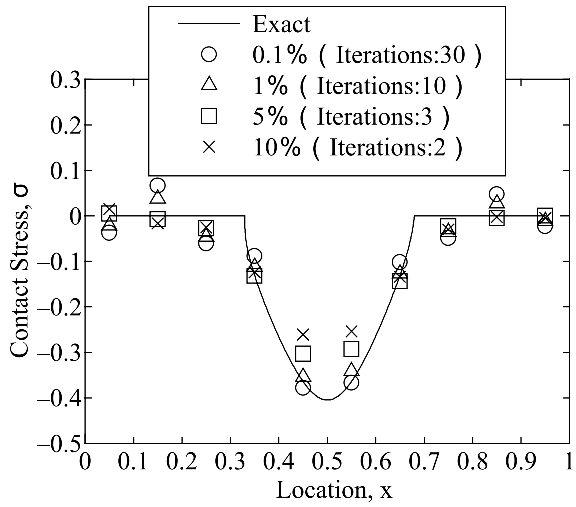

Figure 7: Contact stress distributions estimated by using discrepancy principle applied in observation equations (stress type:Hertzian, n=10, h=0.5, n_{in}=6, h_{in}=0.4, n_{over}=4).

Figures 6 and 7 show the estimated contact stress distributions for the number of iterations estimated using the discrepancy principle applied in the observation space and in the observations equations, respectively. The estimated distributions are deteriorated with increase in the noise level. The estimated contact stress distributions for the stopping number of iterations estimated using the discrepancy principle applied in the observation space is reasonable, while its peak value is lower than the actual. The estimated contact stress distributions for the number of iterations estimated using the discrepancy principle applied in the observation equations agrees well with the actual one, and the peak value of the estimated distribution is close to the actual one, on the other hand fluctuation of the distribution is observed in the regions of $x < 0.3$ and $x > 0.7$, where no contact stress applies. Non-positiveness of the contact stress may be effectively incorporated to obtain a better estimate.

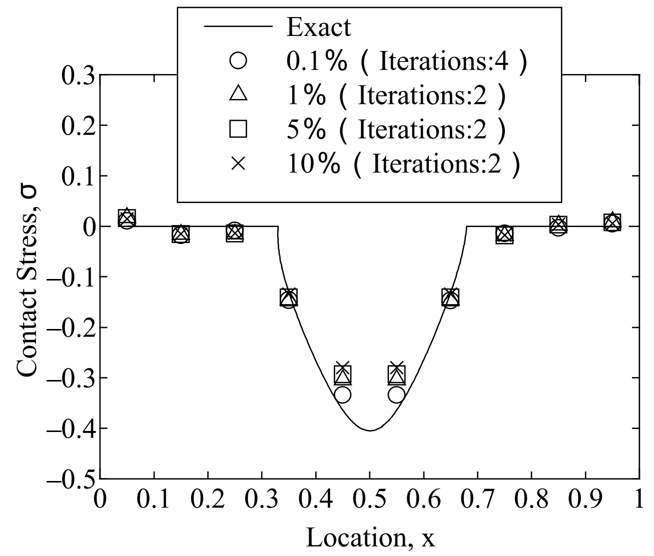

Figure 8: Contact stress distributions estimated by using discrepancy principle applied in observational space (stress type:Hertzian, n=10, h=0.5, n_{in}=9, h_{in}=0.4, n_{over}=1).

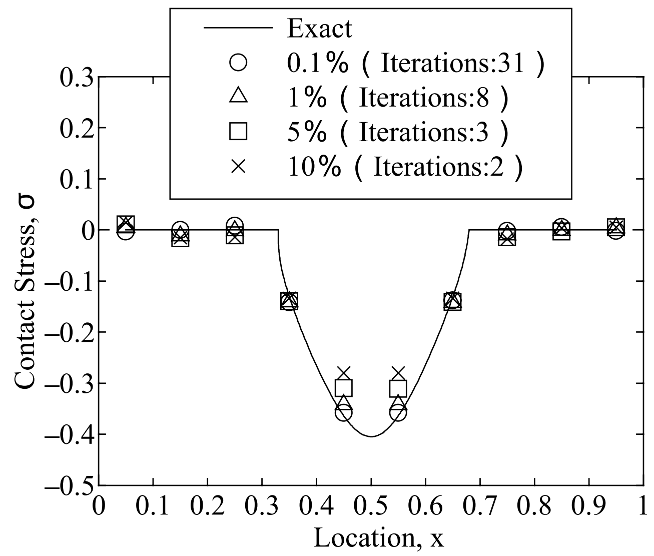

Figure 9: Contact stress distributions estimated by using discrepancy principle applied in observation equations (stress type:Hertzian, n=10, h=0.5, n_{in}=9, h_{in}=0.4, n_{over}=1).

Figures 8 and 9 show the estimated contact stress distributions for the number of iterations estimated using the discrepancy principle in observation space and observations equations, respectively, for the case of h=0.5, h_{in}=0.4, n_{in}=9, and n_{over}=1. It is seen that the contact stress distribution is reasonably estimated for this case also by the present alternating inverse analysis scheme.

7. CONCLUSIONS

Boundary values for a two-dimensional elastostatic body are estimated using the inside observations of displacements together with boundary observations. Alternating boundary element inverse analysis scheme was developed for this situation. The discrepancy principle applied in the observation space and in the observations equations were used to estimate the stopping number of iterations, which is regarded as a regularization parameter. It was found that the alternating boundary element inverse analysis scheme was applicable for the estimation of boundary values from the inside and boundary observations.

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